SOCIAL CONSTRUCTIVISM AS A PHILOSOPHY OF MATHEMATICS: RADICAL CONSTRUCTIVISM REHABILITATED?

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Epistemological issues, although controversial, are central to teaching and learning and have long been a theme of PME. A central epistemological issue is that of the philosophy of mathematics. It is argued that the traditional absolutist philosophies need to be replaced by a conceptual change view of mathematics. Building on the principles of radical constructivism together with the assumption of the existence of the physical and social worlds, a social constructivist philosophy of mathematics is proposed. This suggests an explanation of both the apparent objectivity and the utility of mathematics. A consequence is that the criticism that radical constructivism is necessarily solipsistic is overcome.

It is widely recognised that all practice and theories of learning and teaching rest on an epistemology, whether articulated or not. As Rene Thom puts it, for mathematics:

> "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics."
> (Thom 1973, page 204)

Such issues are a recurring theme in PME, which is not surprising since Piaget, who might be named the honorary god-father of PME, developed probably the most important developmental psychology theory of the century, with epistemological goals explicitly in mind. Of course it is not only the theory and practice of teaching and learning that rests on an epistemology, but also the theory and practice of educational and scientific research. For naturally epistemological issues concern not only the subject matter into which PME inquires, but also the methods by which it carries out and validates its research.

It is to the credit of PME that it is continually seeking to explore its theoretical and philosophical foundations. This contrasts with many other professional organisations, say in medicine, who not only fail to question their epistemological basis, but who also assume without question its consequences, namely certain traditional methods of inquiry. PME is not so complacent, asking such questions as Why is an epistemological perspective a necessity for research in mathematics education? (PME-NA 1983) What is constructivism? What are its implications? (PME11) What is research? (PME13).

However, there is a further dimension to this discussion which needs to be explicitly acknowledged. Namely, that the issues are controversial, and generate strong opinions and feelings in debate. von Glasersfeld makes this point most eloquently.

> "To introduce epistemological considerations into a discussion of education has always been dynamite. Socrates did it, and he was promptly given hemlock. Giambattista Vico did it in the 18th century, and the philosophical establishment could not bury him fast enough."von Glasersfeld (1983, page 41)

Epistemology is controversial, and it cannot be denied that controversy leads to argument and conflict, which is uncomfortable. It is also contrary to the conventional social morality, which seeks consensus, calm, and above all, stability. Controversy represents a threat to this stability. However, the irony is that we probably all subscribe to a belief in the key role of dissonance and cognitive conflict in the accommodation of schemas, and hence in the growth of personal knowledge. Without psychological conflict of this type the growth of personal knowledge is not possible. Likewise, we probably mostly accept a similar view of the role of conflict - revolution even, following Kuhn (1962) and others - in the growth of objective, scientific knowledge. When a new theory threatens to replace an old one, it is not greatly welcomed by those who have invested their professional lives in developing the old theory. Our belief in the necessity of conflict for the growth of knowledge need not eliminate the sensations of discomfort, whether one is involved in conflict at the level of subjective knowledge, objective knowledge, or of social discourse.

Having acknowledged that epistemological issues are controversial, a corollary follows. Participants evidently have different perspectives and belief systems, even different epistemologies. Conflict and disagreement represent the clash of these different perspectives. Such differences can be conceptual, concerning the meanings of such terms as 'epistemology', 'constructivism' or even 'knowledge'. They can be philosophical differences, concerning such issues as the nature of mathematics and the foundations of mathematical knowledge, or general epistemological questions such as 'What is knowledge?', 'What is research?' A consequence of conflict and heated debate is that participants adopt polarized positions, and ascribe simplified or stereotyped 'straw man' positions to their opponents, or to opposing views. Thus, for example, terming the weaker form of constructivism 'trivial constructivism' is a polemical move, using a value-laden, indeed pejorative term, to denigrate a position in the debate.

In this introduction I attempt to relate the issues treated by this paper to the concerns and history of PME. This shows that there is a continuing tradition of epistemological debate, and it has claim to a central place in PME, conflict notwithstanding. My aims are twofold. First, to build on, acknowledge and extend past work. This should be an explicit aim of any contribution to a scientific organisation. Second, to legitimate my inquiry, and show that it central to the concerns of PME.

CONTROVERSY IN THE PHILOSOPHY OF MATHEMATICS

The fundamental problem of the philosophy of mathematics is that of the status and foundation of mathematical knowledge. What is the basis of mathematical knowledge? What gives it its seeming certainty, and is this certainty justified? Two main currents in the philosophy of mathematics can be distinguished. These may be termed absolutist and conceptual change philosophies of mathematics, following the usage of Confrey (1981). Absolutist philosophies of mathematics, including Logicism, Formalism, Intuitionism and Platonism, assert that mathematics is a body of absolute and certain knowledge. In contrast, conceptual change philosophies assert that mathematics is corrigible, fallible and a changing social product. This second claim is shocking, for mathematics is seen by many to be the last bastion of certainty. Perhaps the most important statement of this claim is to be found in Lakatos (1976), and even here the editors added footnotes repudiating Lakatos' fallibilist philosophy of mathematics. Thus, it must be acknowledged that this is a controversial dichotomy. Whilst in science absolutist views have largely given way to conceptual change views, following the work of Hanson, Kuhn, Lakatos, Feyerabend and others, absolutist philosophies of mathematics are still the dominant view. Absolutists believe that mathematical truths are universal, independent of humankind (mathematics is discovered, not invented), and culture- and value-free.

However, the absolutist view is increasingly subject to challenge and attack, for example by Lakatos (1976, 1978), Davis and Hersh (1980), Kitcher (1983), and Tymoczko (1986), as well as many others including Putnam, Bloor and Wittgenstein (1956). The fallibilist position is gaining acceptance year by year, as is illustrated by the publications of philosophically minded mathematics educators in journals such as For the Learning of Mathematics. In the brief space available I will sketch two criticisms of absolutism.

Proof, via deductive logic, is the means by which the certainty of mathematical knowledge is established. However, absolute certainty cannot be gained in this way. As Lakatos (1978) shows, despite all the foundational work and development of mathematical logic, the quest for certainty in mathematics leads inevitably to an infinite regress. Any mathematical system depends on a set of assumptions, and there is no way of escaping them. All we can do is to minimise them, to get a reduced set of axioms and rules of proof. This reduced set cannot be dispensed with, only replaced by assumptions of at least the same strength. Thus we cannot establish the certainty of mathematics without assumptions, which therefore is conditional, not absolute certainty. Only from an assumed basis do the theorems of mathematics follow.

Given that mathematical knowledge is tentative in this sense, are not the theorems of mathematics certain within the assumed axiomatic system? Again the answer is negative. For to establish that mathematical systems are safe (ie. consistent, and we cannot have certainty without consistency), then Godel's Second Incompleteness Theorem shows that for any but the simplest systems (e.g. weaker than Peano Arithmetic) to prove consistency we must add to the set of assumptions, i.e. rely on the consistency of a larger set of assumptions. Thus any attempt to establish the certainty of mathematical knowledge via deductive logic and axiomatic systems fails, except in trivial cases, but including Intuitionism, Logicism and Formalism.

Disposing of absolutism is all very well, but a replacement philosophy must account for the unique features of mathematical knowledge. In particular: How to account for the apparent certainty and objectivity of mathematical knowledge? How, in Wigner's phrase, to account for 'the unreasonable effectiveness of mathematics' in describing the world, and indeed via science, in giving us an unparalleled power over the natural world?

SOCIAL CONSTRUCTIVISM AS A PHILOSOPHY OF MATHEMATICS

The social constructivist thesis is that mathematics is a social construction, a cultural product, fallible like any other branch of knowledge. This view entails two claims. First of all, the origins of mathematics are social or cultural. This is not controversial, and is convincingly documented by many authors such as Bishop (1988) and Wilder (1981). Secondly, the justification of mathematical knowledge rests on its quasi-empirical basis. This is the controversial view put forward by a growing number of philosophers representing the new wave in the philosophy of mathematics, such as Lakatos (1976, 1978), Davis and Hersh (1980), Kitcher (1983), Tymoczko (1986) and Wittgenstein (1956). For the social constructivist account of mathematics to be minimally adequate it must it must offer satisfactory solutions to the two problems described above.

In order to address these issues two assumptions of social constructivism need to be made explicit. First of all, the assumption of realism - there is an enduring physical world, as our common-sense tells us. Secondly, the assumption of social reality - any discussion, including this one, presupposes the existence of the human race and language (the denial of this is potentially inconsistent). Thus I assume the existence of social and physical reality, without presupposing any certain knowledge of either. These assumptions allow a social constructivist epistemology to be developed from the two principles of radical constructivism, which are

a. " knowledge is not passively received but actively built up by the cognizing subject;
b. the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality."von Glasersfeld (1989, page 182)

With the added assumptions of the existence of social and physical reality I can extend these principles to elaborate the epistemological basis of social constructivism.

c. the personal theories which result from the organization of the experiential world must 'fit' the constraints imposed by physical and social reality;
d. they achieve this by a cycle of theory-prediction-test-failure-accommodation-new theory;
e. this gives rise to socially agreed theories of the world and social patterns and rules of language use;
f. mathematics is the theory of form and structure that arises within language.

This provides the basis for a social constructivist philosophy of mathematics. Its elaboration draws on Wittgenstein's (1956) account of mathematical certainty as based on linguistic rules of use and 'forms of life', and Lakatos' (1976) account of the social negotiation of mathematical concepts, results and theories. The result is a philosophical analogue of Restivo's (1988) sociological account of mathematics as a social construction. There is no space to give a full account of social constructivism (which I do elsewhere, in Ernest, forthcoming). However, if the theory is accepted tentatively, it is possible to indicate how it addresses the two problems described above.

One problem is to account for 'the unreasonable effectiveness of mathematics' in describing the world via the theories of science. First of all, the concepts of mathematics are derived by abstraction from direct experience of the physical world, from the generalisation and reflective abstraction of previously constructed concepts, by negotiating meanings with others during discourse, or by some combination of these means. Thus mathematics is a branch of knowledge which is indissolubly connected with other knowledge, through the web of language. Language enables the formulation of theories about social situations and physical reality. Dialogue with other persons and interactions with the physical world play a key role in refining these theories, which consequently are continually being revised to improve their 'fit'. As a part of the web of language, mathematics thus maintains contact with the theories describing social and physical reality, and hence indirectly, with the physical world. The 'fit' of mathematical structures in areas beyond mathematics is continuously being tested, and mathematics is evolving to provide the patterns and solve the tensions that arise from this modelling enterprise. Thus, 'the unreasonable effectiveness of mathematics' is no miracle of coincidence. It is built in. It derives from the empirical and linguistic origins and functions of mathematics.

To account for the apparent certainty and objectivity of mathematical knowledge I claim first that mathematics rests on natural language, and that mathematical symbolism is a refinement and extension of written language. The rules of logic and consistency which permeate the use of natural language provide the bedrock upon which the objectivity of mathematics rests. Mathematical truths arise from the definitional truths of natural language, acquired by social interaction. For example, we normally agree that nothing is both Red and not-Red, and that P & not-P is false. Likewise, The truths of mathematics are defined by implicit social agreement - shared patterns of behaviour - on what constitute acceptable mathematical concepts, relationships between them, and methods of deriving new truths from old. Mathematical certainty rests on socially accepted rules of discourse embedded in our 'forms of life' (Wittgenstein, 1956).

RADICAL CONSTRUCTIVISM REHABILITATED?

Evidently social constructivism offers the possibility of a philosophy of mathematics which accounts for the objectivity and utility of mathematics, as well as its fallibility and culture-boundedness. The cost is that 'objectivity' is reinterpreted as 'social', and although it still refers to something external to the individual it is no longer external to humankind. Clearly this is controversial. But what this account also offers is an answer to the main criticism directed at radical constructivism, namely that it is a solipsistic (Goldin, 1989) "epistemology that makes all knowing active and all knowledge subjective" (Kilpatrick, 1987, page 10). I have argued that the principles of radical constructivism are consistent with, and can be supplemented by assumptions of the existence of physical and social reality. Thus the denial of the existence of the external world is not entailed. In fact, radical constructivism is ontologically neutral, for no claim is made as to the substratum of experience, only that it is unknowable. The case I have argued is that the subjectivity of mathematical knowledge which seems to follow from the principles of radical constructivism does not (just as Intuitionism is not entailed by them, as Lerman, 1989, shows). On the contrary, additional assumptions can safeguard the objectivity of mathematics, when it is viewed as a social construction.

Finally, a further criticism often directed at radical constructivism is that it does not entail a theory of teaching, let alone being the theory of discovery, problem solving and investigational teaching (Goldin, 1989; Kilpatrick, 1987). Here I must agree. To lay the foundation for a philosophy of mathematics, I have had to add further assumptions to those of constructivism. To derive a theory of teaching I must add yet more, not least of which is a set of values. For education depends on the assumption as to what is valuable in our culture to pass on. No such values are assumed in the two principles of radical constructivism.

REFERENCES

Bishop, A.J. (1988) Mathematical Enculturation, Dordrecht: Kluwer.
Confrey, J. (1981) Conceptual Change Analysis: Implications for Mathematics and Curriculum, Curriculum Inquiry, Vol. 11, No. 5, 243-257.
Davis, P.J. and Hersh, R. (1980) The Mathematical Experience, Harmondsworth: Penguin.
Ernest, P. (1991) Philosophy of Mathematics Education, London: Falmer.
Glasersfeld, E. von (1983) Learning as a Constructive Activity, in Proceedings of PME-NA, Vol.1, 41-69.
Glasersfeld, E. von (1989) Constructivism in Education, in Husen, T. and Postlethwaite, N. Eds. (1989) International Encyclopedia of Education (Supplementary Vol.), Oxford: Pergamon, 162-163.

Goldin, G. (1989) Constructivist Epistemology and Discovery Learning in Mathematics, Proceedings of PME 13, Vol.2, Paris, 15-22.
Kilpatrick, J. (1987) What Constructivism might be in Mathematics Education, in Proceedings of PME 11, Volume 1, Montreal, 3-27.
Kitcher, P. (1983) The Nature of Mathematical Knowledge, Oxford, Oxford University Press.
Kuhn, T. (1962) Structure of Scientific Revolutions, Chicago: Chicago UP
Lakatos, I. (1976) Proofs and Refutations, Cambridge: CUP.
Lakatos, I. (1978) Philosophical Papers, Vol.2, Cambridge: CUP.
Lerman, S. (1989) Constructivism, Mathematics and Mathematics Education, Educational Studies in Mathematics, Vol. 20, 211-223.
Restivo, S. (1988) The Social Construction of Mathematics, Zentralblatt fur Didaktik der Mathematik, Vol. 20, No. 1, 15-19.
Thom, R. (1973) Modern mathematics: does it exist? in Howson, A.G. ed. Developments in Mathematical Education, Cambridge: CUP, 195-209.
Tymoczko, T. ed. (1986) New Directions in the Philosophy of Mathematics, Boston: Birkhauser.
Wilder, R.L. (1981) Mathematics as a Cultural System, Oxford: Pergamon.
Wittgenstein, L. (1956) Remarks on the Foundations of Mathematics, revised edition, Cambridge: MIT Press, 1978.